A Remark on the GOP Algorithm for Global Optimization

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Abstract. In this paper it is shown that a large class of smooth mathematical programming problems can be converted into the standard forms to which the GOP algorithm applies.

Key words. GOP algorithm, D.C. transformation.

Global optimization of nonconvex programming problems has been an important topic in optimization theory and has generated significant interest in recent years. A new primal-relaxed dual method, called the GOP algorithm, is reported to be efficient for bilinear programming problems, polynomial programming problems and rational polynomial programming problems (see, for example, Floudas and Visweswaran, 1990, 1993; and Visweswaran and Floudas, 1992, 1993). The method, however, can be applied only after the problem has been reformulated in the following standard form:

$$\min_{x,y} f(x, y) \qquad \text{(GOP)}$$

$$\text{subject to} \quad g_i(x, y) \leqslant 0\,, \quad h_j(x, y) = c_j\,, \quad x \in X\,, \quad y \in Y$$
$$\text{with} \quad 1 \leqslant i \leqslant k\,, \quad i \leqslant j \leqslant p\,,$$

where X and Y are non-empty compact convex sets in $R^n \times R^m$ $(n, m \geqslant 1)$, $f(\cdot, y)$, $g_i(\cdot, y)$, $f(x, \cdot)$ and $g_i(x, \cdot)$ are differentiable convex functions for any fixed $y \in Y$ or $x \in X$, and $h(x, y)$ is bilinear. Hence this method was not considered applicable to a very broad class of mathematical problems. In this paper we show that a large class of smooth mathematical programming problems can actually be reformulated in this form by a simple transformation of variables.

Let X be a non-empty compact convex set in R^n. Let $F(x)$ and $G_i(x)$ $(1 \leqslant i \leqslant L)$ be continuous functions on X. For sake of simplicity we will assume that $F\ G_i \in C^2(R^n)$. We now consider the following optimization problem:

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$$\min_x F(x), \quad \text{subject } G_i(x) \leqslant 0 \text{ and } x \in X \text{ with } 1 \leqslant i \leqslant L . \qquad \text{(GMP)}$$

It is clear that (GMP) represents a large class of mathematical programming problems. Before giving our result we mention a simple fact in perturbation theory:

LEMMA 1. *Let $A(x)$ be a continuous symmetric $n \times n$ matrix on X. Then there exists $\alpha_0 > 0$ such that $B(x) = \alpha I + A(x)$ is a positive definite matrix on X for $\alpha \geqslant \alpha_0$, where I is the $n \times n$ unit matrix.*

Proof. Note that $B(x)/\alpha = I + A(x)/\alpha$ and that $\|A(x)/\alpha\|_{L^\infty} \to 0$ as $\alpha \to \infty$. Therefore the determinant and all sub-determinants of $B(x)/\alpha$ will converge to the corresponding those of I as $\alpha \to \infty$. Thus there is $\alpha_0 > 0$ such that $B(x)/\alpha$ is positive definite for $\alpha \geqslant \alpha_0$ and so is $B(x)$. □

THEOREM 1. *Let X, F and G_i $(1 \leqslant i \leqslant L)$ satisfy the conditions in (GMP). Then there are functions f, g_i $(1 \leqslant i \leqslant L)$ and h_j $(1 \leqslant j \leqslant n)$ in C^2 $(X \times X)$, satisfying the conditions in (GOP) such that the (GMP) can be equivalently reformulated in the following standard form:*

$$\min_{x,\, y} f(x, y)$$

$$\text{subject to} \quad g_i(x, y) \leqslant 0 , \quad h_j(x, y) = 0 , \quad x \in X \quad \text{and} \quad y \in X ,$$
$$\text{with} \quad 1 \leqslant i \leqslant L , \quad 1 \leqslant j \leqslant n .$$

Proof. Select first an $\alpha > 0$ from Lemma 1 such that $\alpha I + H(F)(x)$ and $\alpha I + H(G_i)(x)$ are positive definite matrices on X, where $H(F)$ and $H(G_i)$ are the Hessian matrices of the functions F and G_i on X. Now let $f(x, y) = F(x) + \alpha x x^T - \alpha x y^T$, $g_i(x, y) = G_i(x) + \alpha x x^T - \alpha x y^T$ for $1 \leqslant i \leqslant L$ and $h_j(x, y) = x_j - y_j$ for $1 \leqslant j \leqslant n$. It follows that f, g_i and h_j satisfy the conditions in (GOP) as for a fixed y the Hessian matrix of f or g_i $(1 \leqslant i \leqslant L)$ is positive definite on X. Moreover it is clear that the problem (GMP) can be equivalently rewritten as

$$\min_{x,y} f(x, y)$$

$$\text{subject to} \quad g_i(x, y) \leqslant 0 , \quad h_j(x, y) = 0 , \quad x \in X \quad \text{and} \quad y \in X$$
$$\text{with} \quad 1 \leqslant i \leqslant L , \quad 1 \leqslant j \leqslant L ,$$

This is the conclusion of the theorem. □

It is important to choose α in the numerical computation. This can be solved by noting that the matrices $H(F)$ and $H(G_i)$ $(1 \leqslant i \leqslant L)$ (the Hessian Matrices of F and G_i) can be decomposed as $\mathrm{QDQ}^{\mathrm{T}}$ and $Q_i D_i Q_i^T$, where D and D_i are the diagonal matrices whose diagonal elements are the eigenvalues of $H(F)$ and $H(G_i)$, and $QQ^T = Q_iQ_i^T = I$. It follows from this fact that α can be chosen as

$$\alpha = -\min_{x \in X} \{0, \lambda(x), \lambda_1(x), \ldots, \lambda_L(x)\} ,$$

where $\lambda(x)$ and $\lambda_i(x)$ are the minimum eigenvalues of $H(F)$ and $H(G_i)$ at x.

A special instance where this theoretical result was applied and α was explicitly obtained is the case of rational polynomials that arise in the structure determination of clusters of atoms and molecules (see Maranas and Floudas, 1992).

Note also that a number of related penalty type transformations that reduce combinatorial problems, bilinear programming, and linear complementarity problems to other forms are reported in chapter 3 of Pardalos and Rosen (1987).

From this result it is clear that the GOP method is actually applicable to very broad mathematical programming problems. All useful finite dimensional problems in practice are virtually covered.

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